Journal of Statistical Physics, Vol. 58, Nos. 3/4, 1990

The Transverse Correlation Length for Randomly Rough Surfaces

A. A. Maradudin[1] and T. Michel[1]

Received July 14, 1989

It is shown by numerical simulations for a random, one-dimensional surface defined by the equation $x_3 = \zeta(x_1)$, where the surface profile function $\zeta(x_1)$ is a stationary, stochastic, Gaussian process, that the transverse correlation length a of the surface roughness is a good measure of the mean distance $\langle d \rangle$ between consecutive peaks and valleys on the surface. In the case that the surface height correlation function $\langle \zeta(x_1)\, \zeta(x_1') \rangle / \langle \zeta^2(x_1) \rangle = W(|x_1 - x_1'|)$ has the Lorentzian form $W(|x_1|) = a^2/(x_1^2 + a^2)$, we find that $\langle d \rangle = 0.9080a$; when it has the Gaussian form $W(|x_1|) = \exp(-x_1^2/a^2)$, we find that $\langle d \rangle = 1.2837a$; and when it has the nonmonotonic form $W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a)$, we find that $\langle d \rangle = 1.2883a$. These results suggest that $\langle d \rangle$ is larger, the faster the surface structure factor $g(|Q|)$ [the Fourier transform of $W(|x_1|)$] decays to zero with increasing $|Q|$. We also obtain the function $P(x_1)$, which is defined in such a way that, if $x_1 = 0$ is a zero of $\zeta'(x_1)$, $P(x_1)\, dx_1$ is the probability that the nearest zero of $\zeta'(x_1)$ for positive x_1 lies between x_1 and $x_1 + dx_1$.

KEY WORDS: Transverse correlation length; rough surfaces.

1. INTRODUCTION

A central role in any theory of randomly rough planar surfaces is played by the surface profile function $\zeta(\mathbf{x}_{\parallel})$, which defines the position of the surface through the equation $x_3 = \zeta(\mathbf{x}_{\parallel})$. Here $\mathbf{x}_{\parallel} = \hat{x}_1 x_1 + \hat{x}_2 x_2$, where $\hat{x}_1$ and $\hat{x}_2$ are unit vectors along the x_1 and x_2 axes, is a position vector in the plane $x_3 = 0$. It is usually assumed that $\zeta(\mathbf{x}_{\parallel})$ is a single-valued function of

[1] Department of Physics and Institute for Surface and Interface Science, University of California, Irvine, California 92717.

0022-4715/90/0200-0485$06.00/0 © 1990 Plenum Publishing Corporation

$\mathbf{x}_{||}$, and a stationary, stochastic, Gaussian process, characterized by the properties

$$\langle \zeta(\mathbf{x}_{||}) \rangle = 0 \tag{1.1a}$$

$$\langle \zeta(\mathbf{x}_{||})\, \zeta(\mathbf{x}'_{||}) \rangle = \delta^2 W(|\mathbf{x}_{||} - \mathbf{x}'_{||}|) \tag{1.1b}$$

In these equations the angular brackets denote an average over the ensemble of realizations of $\zeta(\mathbf{x}_{||})$, and $\delta^2 = \langle \zeta^2(\mathbf{x}_{||}) \rangle$ is the mean-square departure of the surface from flatness. On the basis of the physical assumption that on a randomly rough surface the heights of the surface at two widely seprated points $\mathbf{x}_{||}$ and $\mathbf{x}'_{||}$ are uncorrelated, the correlation function $W(|\mathbf{x}_{||}|)$ is required to vanish as $|\mathbf{x}_{||}| \to \infty$. By definition, it has the value unity at $|\mathbf{x}_{||}| = 0$. Several analytic forms have been used to represent $W(|\mathbf{x}_{||}|)$ in various rough surface studies to date, e.g.,[2]

$$W(|\mathbf{x}_{||}|) = \exp(-x_{||}^2/a^2) \qquad \text{(ref. 1)} \tag{1.2a}$$

$$= \exp(-|\mathbf{x}_{||}|/a) \qquad \text{(ref. 2)} \tag{1.2b}$$

$$= \frac{a^2}{x_{||}^2 + a^2} \qquad \text{(ref. 2)} \tag{1.2c}$$

$$= J_0(2\pi\, |\mathbf{x}_{||}|/a) \qquad \text{(ref. 3)} \tag{1.2d}$$

The characteristic distance over which $W(|\mathbf{x}_{||}|)$ decreases significantly from unity is called the *transverse correlation length.* A precise definition of this quantity does not appear to have been given in the literature. An intuitively appealing definition is that the transverse correlation length b is obtained from

$$\pi b^2 = \int d^2x_{||}\; W(|\mathbf{x}_{||}|) \tag{1.3}$$

This, however, is not a completely satisfactory definition, for when it is applied to the four correlation functions given by Eqs. (1.1a)–(1.1d) it yields

$$b = a \tag{1.4a}$$

$$b = \sqrt{2}\, a \tag{1.4b}$$

$$b = \infty \tag{1.4c}$$

$$b = 0 \tag{1.4d}$$

respectively. It is the fact that b equals the distance a at which the Gaussian correlation function $W(|\mathbf{x}_{||}|) = \exp(-x_{||}^2/a^2)$ has decreased to $1/e$ of its

[2] The authors of ref. 3 denote by k_R the quantity we call $2\pi/a$.

initial value, which prompts the definition (1.3). However, its failure to yield meaningful values for the transverse correlation length for the forms of $W(|\mathbf{x}_{||}|)$ given by Eqs. (1.2c) and (1.2d), both of which contain an explicit characteristic length a beyond which the correlation function decreases more or less rapidly to zero, makes it a less than satisfactory definition of the transverse correlation length. Thus, in common with most workers in the field of surface roughness, we will call the characteristic length a that appears in Eqs. (1.1a)–(1.1d) the transverse correlation length.

Although the transverse correlation lenght is defined as the distance over which the correlation function $W(|\mathbf{x}_{||}|)$ decreases significantly from its value of unity at $|\mathbf{x}_{||}| = 0$, one of our aims in this paper is to show that it also has what may be a more readily visualized interpretation, viz. it is a measure of the average distance between consecutive peaks and valleys on the randomly rough surface. Such a correspondence is of interest because, for example, estimates of the angular width of the peak in the retroreflection direction in the angular distribution of the intensity of light scattered from a randomly rough reflecting surface depend on estimates of this distance.[4]

In this paper we present a demonstration of this correspondence for the simpler case of a random surface whose surface profile function $\zeta(x_1)$ is a function of only one coordinate in the plane of the mean surface $x_3 = 0$, i.e., for a random grating. Such surfaces are of interest because they have been used in experimental studies of the attenuation of Rayleigh surface acoustic waves by surface roughness[5] and of the enhanced backscattering of light from rough metal surfaces.[6] The corresponding demonstration for a two-dimensional, randomly rough surface will be presented elsewhere.

For a one-dimensional randomly rough surface the analogues of Eqs. (1.1) are

$$\langle \zeta(x_1) \rangle = 0 \tag{1.5a}$$

$$\langle \zeta(x_1)\, \zeta(x_1') \rangle = \delta^2 W(|x_1 - x_1'|) \tag{1.5b}$$

where $\delta^2 = \langle \zeta^2(x_1) \rangle$. For what follows it is convenient to introduce the Fourier integral representation of $\zeta(x_1)$,

$$\zeta(x_1) = \int_{-\infty}^{\infty} \frac{dQ}{2\pi}\, \hat{\zeta}(Q)\, e^{iQx_1} \tag{1.6}$$

The Fourier coefficient $\hat{\zeta}(Q)$ is also a Gaussianly distributed random variable, characterized by the properties

$$\langle \hat{\zeta}(Q) \rangle = 0 \tag{1.7a}$$

$$\langle \hat{\zeta}(Q)\, \hat{\zeta}(Q') \rangle = 2\pi\delta(Q + Q')\, \delta^2 g(|Q|) \tag{1.7b}$$

where the *surface structure factor* $g(|Q|)$ is defined by

$$g(|Q|)=\int_{-\infty}^{\infty} dx_1\, W(|x_1|)\, e^{-iQx_1} \tag{1.8}$$

The analogue of Eq. (1.3) for a one-dimensional, randomly rough surface is

$$2b=\int_{-\infty}^{\infty} dx_1\, W(x_1)=g(0) \tag{1.9}$$

but we will not make essential use of this relation here.

If we denote by N_L the number of zeros of $\zeta'(x_1)$ in a segment of the x_1 axis of length L, the average distance between peaks and valleys on the surface for a given realization of the surface is $d=L/N_L$. We will be interested in the value of d averaged over the ensemble of realizations of the surface. Thus, the quantity we wish to calculate is

$$\langle d\rangle=\langle L/N_L\rangle \tag{1.10}$$

in the limit as $L\to\infty$.

2. THE DETERMINATION OF $\langle d\rangle$

We calculated $\langle N_L^{-1}\rangle$ in two different ways. The first is based on the integral representation

$$\frac{1}{N_L}=\int_0^{\infty} dt\, e^{-tN_L} \tag{2.1}$$

so that

$$\begin{aligned}\left\langle\frac{1}{N_L}\right\rangle&=\int_0^{\infty} dt\,\langle e^{-tN_L}\rangle\\&=\int_0^{\infty} dt\exp(\langle e^{-tN_L}-1\rangle_c)\end{aligned} \tag{2.2}$$

where $\langle\cdots\rangle$ denotes the cumulant average.[7] We rewrite Eq. (2.2) as

$$\begin{aligned}\left\langle\frac{1}{N_L}\right\rangle&=\int_0^{\infty} dt\exp\left\{\sum_{n=1}^{\infty}\frac{(-1)^n}{n!}t^n\langle N_L^n\rangle_c\right\}\\&=\int_0^{\infty} dt[\exp(-t\langle N_L\rangle_c)]\left\{1+\frac{1}{2}t^2\langle N_L^2\rangle_c\right.\\&\quad\left.-\frac{1}{6}t^3\langle N_L^3\rangle_c+\frac{1}{24}t^4[\langle N_L^4\rangle_c+3\langle N_L^2\rangle_c^2]+\cdots\right\}\\&=\frac{1}{\langle N_L\rangle_c}\left\{1+\frac{\langle N_L^2\rangle_c}{\langle N_L\rangle_c^2}-\frac{\langle N_L^3\rangle_c}{\langle N_L\rangle_c^3}+\frac{\langle N_L^4\rangle_c+3\langle N_L^2\rangle_c^2}{\langle N_L\rangle_c^4}+\cdots\right\}\end{aligned} \tag{2.3}$$

If we now use the results that[7]

$$\langle N_L \rangle_c = \langle N_L \rangle \tag{2.4a}$$

$$\langle N_L^2 \rangle_c = \langle N_L^2 \rangle - \langle N_L \rangle^2 \tag{2.4b}$$

$$\langle N_L^3 \rangle_c = \langle N_L^3 \rangle - 3\langle N_L^2 \rangle \langle N_L \rangle + 2\langle N_L \rangle^3 \tag{2.4c}$$

$$\langle N_L^4 \rangle_c = \langle N_L^4 \rangle - 4\langle N_L^3 \rangle \langle N_L \rangle - 3\langle N_L^2 \rangle^2 + 12\langle N_L^2 \rangle \langle N_L \rangle^2 - 6\langle N_L \rangle^4 \tag{2.4d}$$

we obtain finally

$$\left\langle \frac{1}{N_L} \right\rangle = \frac{1}{\langle N_L \rangle} \left\{ 1 + \frac{\langle N_L^2 \rangle - \langle N_L \rangle^2}{\langle N_L \rangle^2} - \frac{\langle N_L^3 \rangle - 3\langle N_L^2 \rangle \langle N_L \rangle + 2\langle N_L \rangle^3}{\langle N_L \rangle^3} + \frac{\langle N_L^4 \rangle - 4\langle N_L^3 \rangle \langle N_L \rangle + 6\langle N_L^2 \rangle \langle N_L \rangle^2 - 3\langle N_L \rangle^4}{\langle N_L \rangle^4} + \cdots \right\} \tag{2.5a}$$

$$\equiv \frac{1}{\langle N_L \rangle} \{ 1 + \mu_2 + \mu_3 + \mu_4 + \cdots \} \tag{2.5b}$$

According to Jensen's second inequality (see, e.g., ref. 8)

$$\left\langle \frac{1}{N_L} \right\rangle > \frac{1}{\langle N_L \rangle} \tag{2.6}$$

the expansion in braces on the right-hand side of Eq. (2.5b) must be greater than unity. It is of interest to determine by how much it exceeds unity, because if the sum $\mu_2 + \mu_3 + \mu_4 + \cdots$ is small compared to unity, then it is a good approximation to replace $\langle N_L^{-1} \rangle$ by $\langle N_L \rangle^{-1}$. This is also a useful approximation because, as we will see below, $\langle N_L \rangle$ can be calculated analytically in many cases of interest. In addition, it is of interest to determine how rapidly this expansion converges.

The averages $\langle N_L^n \rangle$ entering Eq. (2.5a) were calculated numerically. For this it was necessary first to compute the surface profile function $\zeta(x_1)$, which is a Gaussianly distributed random variable possessing the properties specified by Eq. (1.5). We did this by defining a set of points $\{x_n\}$ along the x_1 axis by $x_n = n\Delta x$, with $n = 0, \pm 1, \pm 2, \ldots$, where the length Δx will be defined precisely below. We then expressed the surface profile function $\zeta(x_1)$ at $x_1 = x_k$ in the form

$$\zeta(x_k) = \delta \sum_{j=-\infty}^{\infty} W_j X_{j+k} \tag{2.7}$$

where $\{X_n\}$ for for $n=0, \pm 1, \pm 2,...$ is a sequence of independent, Gaussian variables with zero mean and a standard deviation of unity.

$$\langle X_i \rangle = 0 \tag{2.8a}$$

$$\langle X_i X_j \rangle = \delta_{ij} \tag{2.8b}$$

while the $\{W_j\}$ are a set of as yet undetermined weights. From the properties (2.8) we find that

$$\langle \zeta(x_k) \rangle = 0 \tag{2.9a}$$

$$\langle \zeta(x_k)\, \zeta(x_{k+l}) \rangle = \delta^2 \sum_{j=-\infty}^{\infty} W_j W_{j-l} \tag{2.9b}$$

From Eqs. (1.5b) and (1.8) we have

$$\begin{aligned}\langle \zeta(x_k)\, \zeta(x_{k+l}) \rangle &= \delta^2 W(|x_l|) \\ &= \delta^2 \int \frac{dQ}{2\pi} e^{iQx_l} g(|Q|)\end{aligned} \tag{2.10}$$

from which it follows that

$$\sum_{j=-\infty}^{\infty} W_j W_{j-l} = \int \frac{dQ}{2\pi} e^{iQx_l} g(|Q|) \tag{2.11}$$

If we now introduce the representation

$$W_j = \int \frac{dQ}{2\pi} \hat{W}(Q)\, e^{iQx_j} \tag{2.12}$$

and use the result that

$$\sum_{j=-\infty}^{\infty} f(x_j) = \frac{1}{\Delta x} \int dx_1 f(x_1) \tag{2.13}$$

in the limit as $\Delta x \to 0$, we obtain

$$\frac{1}{\Delta x} \int \frac{dQ}{2\pi} \hat{W}(Q)\, \hat{W}(-Q)\, e^{iQx_l} = \int \frac{dQ}{2\pi} g(|Q|)\, e^{iQx_l} \tag{2.14}$$

With the assumption, justified by the results, that $\hat{W}(Q)$ is real, we find that

$$\hat{W}(Q) = (\Delta x)^{1/2} g^{1/2}(|Q|) \tag{2.15}$$

It follows finally that

$$W_j = (\Delta x)^{1/2} \int \frac{dQ}{2\pi} g^{1/2}(|Q|)\, e^{iQx_j} \tag{2.16}$$

Although $\zeta(x_k)$ could be calculated by the use of Eqs. (2.7) and (2.16), it proved advantageous to use a modification of these results that exploits the speed of the fast Fourier transform. This modification consists in assuming that $\zeta(x_k)$ is a periodic function of k with period $2M$, $\zeta(x_{k+2M}) = \zeta(x_k)$, where $2M$ is a large integer. We can obtain a representation of $\zeta(x_k)$ with this property by requiring that W_j and X_j also possess this periodicity, and writing

$$W_j = \frac{1}{(2M)^{1/2}} \sum_{l=-M}^{M-1} \hat{W}_l e^{i(2\pi lj/2M)} \tag{2.17a}$$

$$X_j = \frac{1}{(2M)^{1/2}} \sum_{l=-M}^{M-1} \hat{X}_l e^{i(2\pi lj/2M)} \tag{2.17b}$$

It then follows that if we write

$$\zeta(x_k) \cong \delta \sum_{j=-M}^{M-1} W_j X_{j+k} \tag{2.18}$$

we obtain the result that

$$\zeta(x_k) = \delta \sum_{l=-M}^{M-1} \hat{W}_{-l} \hat{X}_l e^{i(2\pi lk/2M)} = \zeta(x_{k+2M}) \tag{2.19}$$

From the inverse to Eq. (2.17a)

$$\hat{W}_l = \frac{1}{(2M)^{1/2}} \sum_{j=-M}^{M-1} W_j e^{-i(2\pi lj/2M)} \tag{2.20}$$

we obtain the approximation

$$\hat{W}_l = \frac{1}{(2M)^{1/2}\,\Delta x} \int_{-\infty}^{\infty} dx\, W(x)\, e^{-1(2\pi lx/\mathscr{L})} \tag{2.21}$$

in the limit as $M \to \infty$, $\Delta x \to 0$, while $\mathscr{L} = 2M\Delta x$. With the use of Eq. (2.16), this becomes

$$\hat{W}_l = \frac{1}{(2M\Delta x)^{1/2}} g^{1/2}(|q_l|) \tag{2.22}$$

where $q_l \equiv 2\pi l/\mathscr{L}$.

We next use the inverse to Eq. (2.17b) to write

$$\hat{X}_l = \frac{1}{\sqrt{2}}(M_l + iN_l) \tag{2.23}$$

where

$$M_l = \frac{1}{\sqrt{M}} \sum_{j=-M}^{M-1} X_j \cos\frac{2\pi lj}{2M} = M_{-l} \tag{2.24a}$$

$$N_l = -\frac{1}{\sqrt{M}} \sum_{j=-M}^{M-1} X_j \sin\frac{2\pi lj}{2M} = -N_{-l} \tag{2.24b}$$

It follows that[3]

$$\zeta(x_k) = \frac{\delta}{(2M\,\Delta x)^{1/2}} \sum_{l=-M}^{M-1} \frac{1}{\sqrt{2}}(M_l + iN_l)\, g^{1/2}(|q_l|)\, e^{iq_l x_l} \tag{2.25}$$

The representation for $\zeta(x_k)$ given by Eq. (2.25) is convenient not only because it is in a form that can be readily evaluated by the fast Fourier transform, but also because from the definitions (2.24) and the properties (2.8) it follows that the $\{M_l\}$ and $\{N_l\}$ are themselves independent, Gaussian variables with zero mean and a standard deviation of unity. They can be generated by the Marsaglia and Bray modification of the Box–Muller transformation of a pair of uniform deviates between zero and one obtained from a linear congruential generator.[(10)]

The derivative of the surface profile function evaluated at $x_1 = x_k$ is obtained by differentiation of the expression given by Eq. (2.25) with respect to x_k:

$$\zeta'(x_k) = \frac{\delta}{(2M\,\Delta x)^{1/2}} \sum_{l=-M}^{M} \frac{1}{\sqrt{2}}(M_l + iN_l)\, iq_l g^{1/2}(|q_l|)\, e^{iq_l x_k} \tag{2.26}$$

In practice $\zeta(x_1)$ and $\zeta'(x_1)$ were calculated from Eq. (2.26) at a set of N consecutive values of x_1 defined by $x_1 = x_k$ with $k = -N/2$, $-N/2 + 1, \ldots, N/2 - 1$, which define a segment of the x_1 axis of length $L = N\,\Delta x$. We chose for L the value $L = \frac{1}{2}\mathscr{L}$, i.e., $N = \frac{1}{2}M$. For each realization of $\zeta'(x_1)$ the number of zeros N_L it possessed in this interval was calculated from the number of sign changes in the sequence of values of $\zeta'(x_k)$ as k was swept through the range $(-N/2, N/2 - 1)$. In these calculations N was taken to be 16,384. The calculation of N_L was repeated for each of N_p realizations of $\zeta'(x_1)$, and the results were used in obtaining the

[3] A representation of $\zeta(x_k)$ of the form given by Eq. (2.25) in the particular case $W(|x_1|) = \exp(-x_1^2/a^2)$ was used by Thorsos.[(9)]

averages $\langle N_L^n \rangle$ required for the evaluation of the right-hand side of Eq. (2.5). The angular brackets here denote an average over the N_p realizations of $\zeta(x_1)$ [and of $\zeta'(x_1)$]. In practice, N_p was taken to be 50,000.

We note, parenthetically, that a somewhat similar numerical algorithm has been constructed recently to generate surface profile functions that are stationary, stochastic, but non-Gaussian processes.[11] It yields results whose accuracy is quite comparable to that of the results to be described in the next section.

3. RESULTS

We have calculated $\langle N_L^{-1} \rangle$ by the methods described in the preceding section for three forms for the correlation function $W(|x_1|)$. These, together with the corresponding surface structure factors $g(|Q|)$, are:

(a) $$W(|x_1|) = a^2/(x_1^2 + a^2) \tag{3.1a}$$

$$g(|Q|) = \pi a e^{-|Q|a} \qquad (b = \pi a/2) \tag{3.1b}$$

(b) $$W(|x_1|) = e^{-x_1^2/a^2} \tag{3.2a}$$

$$g(|Q|) = \pi^{1/2} a e^{-Q^2a^2/4} \qquad (b = \pi^{1/2}a/2) \tag{3.2b}$$

(c) $$W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a) \tag{3.3a}$$

$$g(|Q|) = a, \qquad |Q| < \pi/a$$
$$= 0 \qquad |Q| > \pi/a \qquad (b = a/2) \tag{3.3b}$$

The values of b defined by Eq. (1.9) are presented in parentheses for each of the forms of $W(|x_1|)$ used here. However, in what follows the transverse correlation length corresponding to each of these choices will be understood to be the length a that appears in Eqs. (3.1a), (3.2a), and (3.3a).

For each of these forms of $W(|x_1|)$ we calculated $\zeta(x_1)$ and $\zeta'(x_1)$ by the method described in the preceding section. In these calculations Δx was chosen to be $a/40$. As a check on the algorithm used in these calculations, we calculated the correlation function $\langle \zeta(x_k)\, \zeta(x_{k+l}) \rangle / \delta^2 \equiv W(|x_l|)_s$ for each of these forms, and compared the results with the exact expressions for $W(|x_1|)$ [$\equiv W(|x_1|)_t$] given by Eqs. (3.1)–(3.3). The differences between the computed values of these correlation functions and the exact expressions given by Eqs. (3.1)–(3.3) were very small. To show them clearly, in Fig. 1 we have plotted the ratio $[W(|x_1|)_s - W(|x_1|)_t]/[W(|x_1|)_s + W(|x_1|)_t]$. The agreement between the computed and exact results is seen to be excellent for $0 < x_1/a \lesssim 2.5$ in each case. We conclude that our algorithm for computing $\zeta(x_1)$ is sufficiently accurate for our purposes.

We present in Table I our computed values of $\langle n_L^n \rangle$ for $n = -1$, 1, 2, 3, 4, where $n_L = (a/L) N_L$. We note from Eq. (1.10) that $\langle d \rangle = \langle n_L^{-1} \rangle a$. We have also tabulated there the values of the quantities μ_2, μ_3, μ_4, which enter Eq. (2.5b) for $\langle N_L^{-1} \rangle$. From the results presented in this table we see immediately that the two methods for calculating $\langle N_L^{-1} \rangle$

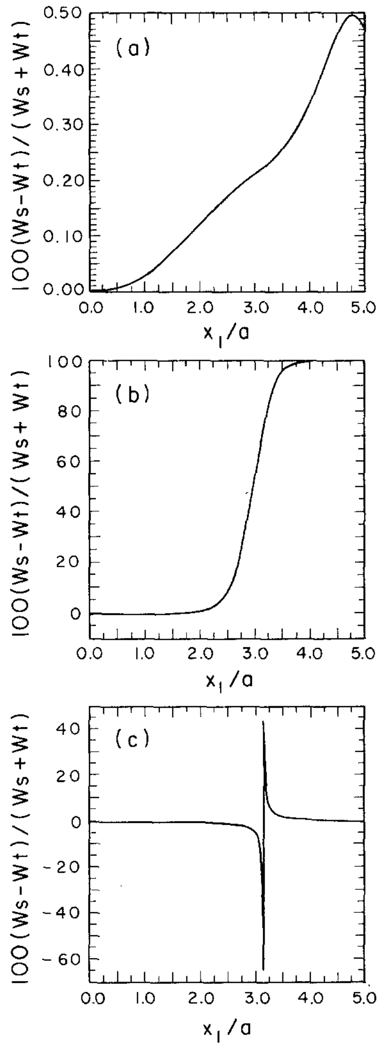

Fig. 1. The ratio $[W(|x_1|)_s - W(|x_1|)_t]/[W(|x_1|)_s + W(|x_1|)_t]$, where $W(|x_1|)_s$ is the computed value of the surface height correlation function and $W(|x_1|)_t$ is the exact expression for it, as a function of x_1/a. (a) $W(|x_1|) = a^2/(x_1^2 + a^2)$; (b) $W(|x_1|) = \exp(-x_1^2/a^2)$; (c) $W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a)$.

Table I. Computed Averages of Powers of $n_L = (a/L)\, N_L$, Where N_L is the Number of Zeros of $\zeta'(x_1)$ in a Segment of the x_1 Axis of Length L, and a is the transverse Correlation Length

	(a)	(b)	(c)
$\langle n_L^{-1}\rangle$	0.907999	1.28374	1.288286
$\langle n_L\rangle$	1.10241	0.779728	0.77806
$\langle n_L^2\rangle$	1.21650	0.608563	0.606670
$\langle n_L^3\rangle$	1.34372	0.475429	0.474129
$\langle n_L^4\rangle$	1.48569	0.371778	0.371372
μ_2	9.83653 (−4)	9.65070 (−4)	2.27225 (−3)
μ_3	8.09697 (−7)	6.34091 (−7)	6.78204 (−6)
μ_4	2.88534 (−6)	2.80194 (−6)	1.54756 (−5)
$\langle n_L^{-1}\rangle_{\text{series}}$	0.907999	1.28374	1.288286
$\langle n_L\rangle^{-1}$	0.907104	1.28250	1.285337
$\langle n_L\rangle_{\text{exact}}$	$12^{1/2}/\pi = 1.102658$	$6^{1/2}/\pi = 0.779697$	$(3/5)^{1/2} = 0.774597$

[a] (a) $W(|x_1|) = a^2/(x_1^2 + a^2)$; (b) $W(|x_1|) = \exp(-x_1^2/a^2)$; (c) $W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a)$.

described in the preceding section, viz. by means of the series (2.5), and by direct calculation of this average, yield results which agree to six significant figures. We see, moreover, that the expansion (2.5) appears to have an asymptotic nature, in that, for each form assumed for $W(|x_1|)$, $\mu_3 < \mu_2 < 1$, but $\mu_4 > \mu_3$. Finally, we see that $\langle d\rangle$ is remarkably well approximated by $a/\langle n_L\rangle$. The percentage differences between $a\langle n_L^{-1}\rangle$ and $a/\langle n_L\rangle$ are 0.099 %, 0.097 %, and 0.229 % for the correlation functions (3.1), (3.2), and (3.3), respectively. We also note that $\langle n_L^{-1}\rangle > \langle n_L\rangle^{-1}$, in agreement with Eq. (2.6).

The accuracy of these calculations can be checked, because $\langle N_L\rangle$ can be calculated exactly for each of the correlation functions (3.1)–(3.3). The number of zeros of $\zeta'(x_1)$ in the interval $0 < x_1 < L$ is given by

$$\begin{aligned} N_L &= \int_0^L dx_1\, |\zeta''(x_1)|\, \delta(\zeta'(x_1)) \\ &= \int_0^L dx_1 \sum_i \delta(x_1 - x_1(i)) \\ &= \sum_i [\theta(L - x_1(i)) - \theta(-x_1(i))] \end{aligned} \tag{3.4}$$

where $x_1(i)$ is the ith zero of $\zeta'(x_1)$, and $\theta(x_1)$ is the Heaviside unit step function. Each zero in the interval $(0, L)$ contributes unity to the sum. The average number of zeros in $(0, L)$ is therefore

$$\langle N_L\rangle = \int_0^L dx_1\, \langle |\zeta''(x_1)|\, \delta(\zeta'(x_1))| \rangle \tag{3.5}$$

We use the representations

$$|x| = \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k)\, e^{ikx} \tag{3.6}$$

where

$$f(k) = \lim_{\varepsilon \to 0+} \frac{2(\varepsilon^2 - k^2)}{(\varepsilon^2 + k^2)^2} \tag{3.7}$$

and

$$\delta(x) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ikx} \tag{3.8}$$

to write

$$\langle |\zeta''(x_1)|\, \delta(\zeta'(x_1))\rangle$$
$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k) \int_{\infty}^{\infty} \frac{dq}{2\pi} \langle e^{ik\zeta''(x_1)} e^{iq\zeta'(x_1)}\rangle \tag{3.9}$$

If we now introduce the Fourier integral representation of $\zeta(x_1)$ given by Eq. (1.6), the average we must evaluate becomes

$$A \equiv \langle e^{ik\zeta''(x_1)} e^{iq\zeta'(x_1)}\rangle$$
$$= \left\langle \exp \int_{-\infty}^{\infty} \frac{dQ}{2\pi} (-ikQ^2 - qQ)\, \hat{\zeta}(Q)\, e^{iQx_1} \right\rangle \tag{3.10}$$

We evaluate this average by cumulant methods,[7] and use the fact that the cumulant averages of higher than second order of a Gaussianly distributed random variable all vanish.[4] With the aid of Eq. (1.7b) the result is

$$A = \exp\left\{ -\frac{1}{2} \delta^2 \left(q^2 \frac{m_2}{a^2} + k^2 \frac{m_4}{a^4} \right) \right\} \tag{3.11}$$

where we have introduced the definition

$$\int_{-\infty}^{\infty} \frac{dQ}{2\pi} g(|Q|)\, Q^{2p} = \frac{m_{2p}}{a^{2p}} \tag{3.12}$$

[4] This result is straightforward to establish by the use of results presented in ref. 7.

The approach to the determination of $\langle N_L \rangle$ described here therefore requires that m_2 and m_4 be finite. This places restrictions on $g(|Q|)$ and hence on $W(|x_1|)$. When Eq. (3.11) is substituted into Eq. (3.9) we find that

$$\begin{aligned}
&\langle |\zeta''(x_1)|\,\delta(\zeta'(x_1))\rangle \\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi} f(k)\, e^{-(m_4\delta^2/2a^4)k^2} \int_{-\infty}^{\infty} \frac{dq}{2\pi}\, e^{-(m_2\delta^2/2a^2)q^2} \\
&= \left(\frac{2}{m_2\pi^3}\right)^{1/2} \frac{a}{\delta} \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} \int_0^{\infty} du \frac{1-u^2}{(1+u^2)^2}\, e^{-(m_4\delta^2\varepsilon^2/2a^4)u^2} \\
&= \frac{1}{a}\left(\frac{m_4}{m_2}\right)^{1/2} \frac{1}{\pi}
\end{aligned} \tag{3.13}$$

The result that this average is independent of x_1 is due to the stationarity of $\zeta(x_1)$. When we combine Eqs. (3.13) and (3.5) we obtain

$$\langle N_L \rangle = \frac{L}{a}\left(\frac{m_4}{m_2}\right)^{1/2} \frac{1}{\pi} \tag{3.14}$$

We remark that the expression (3.14) for $\langle N_L \rangle$ can be extracted from some of the results obtained in the review article by Rice.[12] We believe the derivation presented here is more direct and simpler.

The values of m_2 and m_4 for each of the three expressions for $g(|Q|)$ given by Eqs. (3.1)–(3.3) are

$$\begin{aligned}
&\text{(a)} \qquad m_2 = 2, \qquad m_4 = 24 \\
&\text{(b)} \qquad m_2 = 2, \qquad m_4 = 12 \\
&\text{(c)} \qquad m_2 = \tfrac{1}{3}, \qquad m_4 = \tfrac{1}{5}
\end{aligned} \tag{3.15}$$

The resulting exact values of $\langle N_L \rangle$ are presented in Table I, together with the numerical values computed by the method described in the preceding section. It is seen that the agreement is excellent. The calculated value of $\langle N_L \rangle$ departs fromthe exact value by 0.23 %, −0.0040 %, and −0.44 % for the height correlation functions given by Eqs. (3.1), (3.2), and (3.3), respectively. These results give us additional confidence in the accuracy of the numerical methods used in the computation of $\langle N_L^{-1} \rangle$.

We note that in both the calculation of $\langle \zeta(x_1)\,\zeta(x_1') \rangle/\delta^2$ and of $\langle N_L \rangle$, the poorest accuracy, for the same computational effort, is obtained for the surface height correlation function given by $W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a)$. This is presumably due to the nonmonotonic dependence of this function on x_1.

We remark that in principle averages of higher powers of N_L could also be evaluated in the manner just described, but in practice the resulting calculations are very complicated.

Although the emphasis in this paper is on the mean distance $\langle d \rangle$ between consecutive zeros of the derivative of the surface profile function

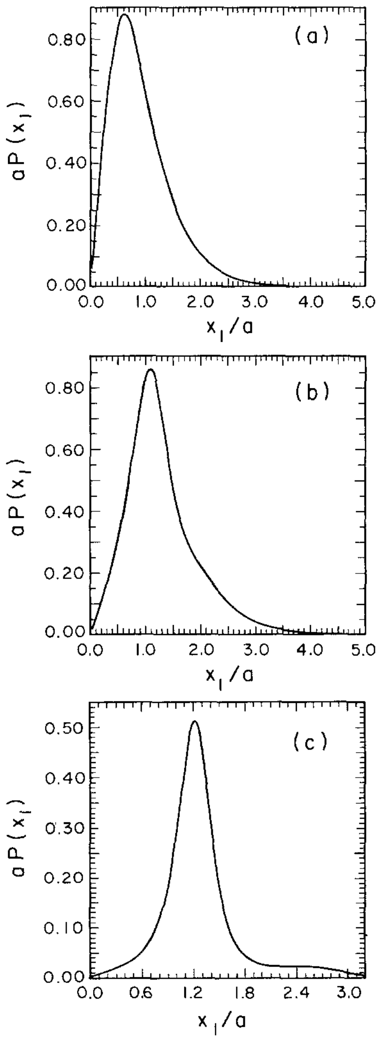

Fig. 2. The function $P(x_1)$, which is defined in such a way that if $x_1 = 0$ is a zero of $\zeta'(x_1)$, $P(x_1)\,dx_1$ is the probability that the nearest zero of $\zeta'(x_1)$ for positive x_1 lies between x_1 and $x_1 + dx_1$. (a) $W(|x_1|) = a^2/(x_1^2 + a^2)$; (b) $W(|x_1|) = \exp(-x_1^2/a^2)$; (c) $W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a)$.

$\zeta'(x_1)$, and its relation to the transverse correlation length a, we note that this distance is but the first moment of the probability distribution function $P(x_1)$, which is defined in such a way that if $x_1 = 0$ is a zero of $\zeta'(x_1)$, then $P(x_1)\,dx_1$ is the probability that the nearest zero of $\zeta'(x_1)$ for positive x_1 lies between x_1 and $x_1 + dx_1$:

$$\langle d \rangle = \int_0^\infty dx_1 x_1 P(x_1) \tag{3.16}$$

The determination of $P(x_1)$ analytically is a very difficult problem, which apparently has not been solved yet.[12] It can, however, be computed easily by the numerical methods used in this paper.

To calculate $P(x_1)$, we first calculated $N_p = 50{,}000$ relizations of $\zeta'(x_1)$ by the method described in the preceding section, at N values of $x_k = k\,\Delta x$ obtained by sweeping k through the range $(-N/2, N/2-1)$. In these calculations N was again chosen to be 16,384, while the period $\mathscr{L}$ of $\zeta(x_1)$, given by $\mathscr{L} = 2M\,\Delta x$, was defined by choosing $M = 2N$. For each realization of $\zeta'(x_1)$ the sequence $\zeta'(x_k)\,\zeta'(x_{k+1})$ was formed for $k = -N/2, \ldots, N/2-1$, and the sign of each product recorded. A negative sign for $\zeta'(x_k)\,\zeta'(x_{k+1})$ indicated that $\zeta'(x_1)$ had a zero between x_k and x_{k+1}, i.e., between x_k and $x_k + \Delta x$. The distances between consecutive zeros were also recorded. Finally, a histogram was constructed from all the data collected that gave the number of times the distance between consecutive zeros of $\zeta'(x_1)$ fell in the interval $(x_k, x_k + \Delta x)$ with $k = 0, 1, 2, \ldots$. Normalized to unity, this histogram yielded $P(x_1)$. In these calculations Δx was again chosen to be $a/40$.

The results of our determinations of $P(x_1)$ for each of the surface height correlation functions given by Eqs. (3.1)–(3.3) are presented in Fig. 2. As expected, this function has a pronounced peak in the vicinity of $x_1 = \langle d \rangle$, which decreases to zero rapidly as x_1 departs from this value. With $P(x_1)$ in hand, one can calculate higher moments of this distribution function than the first. One can extend such calculations to obtain the probability distribution function for the distance between consecutive maxima or consecutive minima on the randomly rough surface. We have not carried out such calculations here, and leave them to subsequent work.

4. DISCUSSION AND CONCLUSIONS

In this work we have described a numerical algorithm for generating a surface profile function $\zeta(x_1)$, and its derivative $\zeta'(x_1)$, that is a stationary stochastic, Gaussian process, characterized by a specified surface height correlation function. We have used it, together with analytic checks on its accuracy, to relate the average separation between consecutive peaks and valleys on the randomly rough surface $\langle d \rangle$ to the transverse correlation length a, which characterizes the spatial decay of the surface height correlation function. We have also used this algorithm in a calculation of the probability distribution function of the separation between consecutive peaks and valleys on the surfaces.

From the results presented in Table I we see that $\langle d \rangle \simeq a$, as stated in the Introduction. Indeed, when the surface height correlation function is given by $W(|x_1|) = a^2/(x_1^2 + a^2)$, we have that $\langle d \rangle = 0.9080a$; when $W(|x_1|) = \exp(-x_1^2/a^2)$, $\langle d \rangle = 1.2837a$; and when $W(|x_1|) = \sin(\pi x_1/a)/(\pi x_1/a)$, $\langle d \rangle = 1.2883a$. We do not pretend that results obtained for three different forms for the correlation function $W(|x_1|)$, albeit of widely differing analytic forms, constitute a proof of this relation for an arbitrary form for $W(|x_1|)$. In the absence of a generally accepted definition of the transverse correlation length, such a proof is not possible. Nevertheless, these results indicate that the transverse correlation length is indeed a good measure of the average distance between consecutive peaks and valleys on the randomly rough surface. They also suggest that $\langle d \rangle$ is larger the faster the surface structure factor $g(|Q|)$ decays to zero with increasing $|Q|$. This is consistent with the increase in $\langle N_L \rangle$ as $g(|Q|)$ decays more slowly with increasing $|Q|$, and the near equality of $\langle N_L^{-1} \rangle$ with $\langle N_L \rangle^{-1}$.

It is, perhaps, not surprising that the transverse correlation length a should be comparable with the mean distance between consecutive zeros of $\zeta'(x_1)$, $\langle d \rangle$, i.e., the distance between consecutive maxima and minima of the surface profile function $\zeta(x_1)$. On one hand, a is the only length characterizing the surface roughness along the surface, i.e., along the x_1 axis. On the other hand, a is, crudely, the distance beyond which the heights of the surface become statistically uncorrelated, while $\langle d \rangle$ is on average the maximum distance over which the surface profile function decreases from a maximum before it starts to increase toward the next maximum. It would be surprising if the latter distance were significantly different from the former.

It is hoped that the results presented here may be helpful to experimentalists engaged in measuring the statistical properties of randomly rough surfaces in refining their estimates of the transverse correlation length a by relating it to a simple geometrical property of a random surface. We also hope that the discussion presented here will stimulate efforts to define the transverse correlation function more precisely. Finally, we would be very pleased if mathematicians were stimulated by these results to attack again the problem of determining the probability distribution function $P(x_1)$ by more analytic methods than were employed here. Theoretical and experimental studies of properties of randomly rough surfaces would be aided by these developments.

ACKNOWLEDGMENTS

We dedicate this paper with esteem and affection to Prof. Cyril Domb, F.R.S., with whom the senior author (A.A.M.) did some of his earliest work on properties of random systems, and to wish him all the best in what will undoubtedly be an active and happy retirement.

We are grateful for an award of time on the CRAY XMP-48 at the Ballistic Research Laboratory of the Aberdeen Proving Ground, Maryland. This work was supported in part by Army Research Office grant DAAL-88-K-0067.

REFERENCES

1. H. Davies, *Proc. Inst. Elect. Eng.* (London) **101**(IV):209 (1954).
2. J. Crowell and R. H. Ritchie, *J. Opt. Soc. Am.* **60**:794 (1970).
3. E. Kretschmann, T. L. Ferrell, and J. C. Ashley, *Phys. Rev. Lett.* **42**:1312 (1979).
4. A. A. Maradudin, E. R. Méndez, and T. Michel, in *Scattering from Volumes and Surfaces*, J. C. Dainty and M. Nieto-Vesperinas, eds. (North-Holland, Amsterdam, 1989).
5. M. deBilly, G. Quentin, and E. Baron, *J. Appl. Phys.* **61**:2140 (1987).
6. J. C. Dainty, M.-J. Kim, and A. J. Sant, Measurements of enhanced backscattering of light from one and two dimensional random rough surfaces, Notes for the Workshop "Recent Progress in Surface and Volume Scattering," Madrid, September 14–16, 1988, and for the International Working Group Meeting on "Wave Propagation in Random Media," Tallinn, September 19–23, 1988.
7. R. Kubo, *J. Phys. Soc. Japan* **17**:1100 (1962).
8. M. Schlesinger, in *Statistical Mechanics and Statistical Methods in Theory and Application*, U. Landman, ed. (Plenum Press, New York, 1977), p. 507.
9. E. I. Thorsos, *J. Acoust. Soc. Am.* **83**:78 (1988).
10. A. Stuart and J. Keith Ord, *Kendall's Advanced Theory of Statistics* (Charles Griffin, London, 1987), pp. 308–309.
11. A. A. Maradudin, E. R. Méndez, and T. Michel, *J. Opt. Soc. Am. B* (in press).
12. S. O. Rice, in *Selected Papers on Noise and Stochastic Processes*, Nelson Wax, ed. (Dover, New York, 1965), p. 133.